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High Order Correction Terms for The Peak-Peak Correlation Function in Nearly-Gaussian Models

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ABSTRACT

Context. One possible way to investigate the nature of the primordial power spectrum fluctuations is by investigating the statistical properties of the local maximum in the density fluctuation fields.
Aims. In this work we present a study of the mean correlation function, ξ_r, and the correlation function for high amplitude fluctuations (peak-peak correlation) in a slighlty non-Gaussian context.
Methods. From the definition of the correlation excess, we compute the Gaussian two-point correlation function and, using an expansion in Generalized Hermite polynomials, we estimate the correlation of high density peaks in a non-Gaussian field with generic distribution and power spectrum. We also apply the results obtained to a scale-mixed distribution model, which correspond to a nearly Gaussian model.
Results. The results reveal that, even for a small deviation from Gaussianity, we can expect high density peaks to be much more correlated than in a Gaussian field with the same power spectrum. In addition, the calculations reveal how the amplitude of the peaks in the fluctuations field is related to the existing correlations.
Conclusions. Our results may be used as an additional tool to investigate the behavior of the N-point correlation function, to understand how non-Gaussian correlations affect the peak-peak statistics and extract more information about the statistics of the density field.

Key words. fluctuations fields – random variable – correlation function

1. Introduction

Investigation of the statistical properties of cosmological density fluctuations is a very useful tool to understanding the origin of the cosmic structure. Roughly, cosmological models to describe primordial fluctuations can be divided in two classes: Gaussian and non-Gaussian. The most accepted model for structure formation assumes initial quantum fluctuations created during inflation and amplified by gravitational effects. The standard inflationary models predicts an uncorrelated random field, with a scale-invariant power spectrum, which follows a nearly-Gaussian distribution (Guth & PI 1982; Gangui et al. 1994). However, non-Gaussian fluctuations are also allowed in a wide class of alternative models, such as: the multiple interactive fields (Allen et al. 1987; Salopek et al. 1989), the cosmic defects models (Kibble 1976) and the hybrid models (Magueijo & Brandenbergher 2000; Battye & Weller 2000). By discriminating between different classes of models, the statistical properties of the fluctuations field can be used to investigate the nature of cosmic structure. However, non-Gaussian models comprehend an infinite range of possible statistics. As a consequence, performing statistical tests of this kind are not a straightforward task, since there is no adequate general test for every kind of model. To attack this problem, any effort to better understand how the statistical properties of the density fluctuation field affect the observed Universe is welcome, since it may bring extra pieces of information to the investigation of cosmic structure.

Due to the great importance of characterizing non-Gaussian signatures, many statistical approaches have been used to study the distribution of fluctuations in the cosmic microwave background radiation (CMBR) (Chiang et al. 2003; Bond & Efstathiou 1987; Eriksen et al. 2005; Cabella et al. 2004; Komatsu et al. 2003) and the large scale structure (LSS) (Frith et al. 2005; 2003; Verde et al. 2001; Fry 1985). One possible way to investigate the nature of the primordial power spectrum fluctuations is by investigating the statistical properties of the local maximum in the density fluctuation fields. Since some of the peak properties, such as number, frequency, correlation, height and extrema, are highly dependent upon the statistics of the fluctuation field, we can gather information about the statistical distribution function by studying the morphological properties of the fluctuations fields (Bardeen et al. 1986;

Novikov et al. 1999; Larson & Wandelt 2005). Other useful statistical estimator applied to investigate density fluctuations are the wavelets tools (McEwen et al. 2004; Mukherjee & Wang 2004; Vielva et al. 2004), the phase correlations (Coles et al. 2004; Naselsky et al. 2005) and the most widely used estimator: the N-point correlations in phase (Lewin et al. 1999; Verde & Heavens 2001; Komatsu et al. 2005) or density spaces (Takada & Jain 2005; Coles & Jones 1991; Heavens & Sketh 1999; Heavens & Gupta 2001; Peebles 2001; Eriksen et al. 2004).

The extensive use of the two-point correlation function to characterize the statistical properties of the fluctuations field is justified by the mathematical simplicity in the Gaussian condition, since the mean correlation function can be obtained analytically and it completely specifies a Gaussian distribution (as well as its power spectrum). However, this assumption is not true for a non-Gaussian case, where higher order correlations may give a significant contribution, despite the great effort demanded to compute a wide range of correlations. However, it is possible to detect primordial non-Gaussianity with a non-zero measure of the N-point correlation ($n \geq 3$). In order to achieve a good description of non-Gaussian signatures in cosmic structure, many works have been done to estimate the three-point correlation function and the related bi or tri-spectrum for a few classes of non-Gaussian models (Verde & Heavens 2001; Komatsu et al. 2003; Eriksen et al. 2005; Komatsu et al. 2005; Gaztanaga & Wagg 2004; Hansen et al. 2004; Creminelli et al. 2005; De Troia et al. 2003). It is believed that, with the advent of the CMBR experiments and the high quality surveys in cosmology, the N-point correlation function will be the main statistical descriptor for the cosmic structure.

In this work we present a technique to extract non-Gaussian components from a two-point correlation function. In presence of a non-Gaussian component, we point out that even between two points it is possible to estimate the influence of higher order correlations for models with different statistical description. By performing the calculation of the high order correction terms for the peak-peak correlation function, we show how the amplitude of peaks is dependent of the correlations involved. We point out that for a non-Gaussian statistics the higher order correlations impose some restrictions to the amplitude of high density peaks.

This paper is divided as follows: in Section 2 we give a general description of a random variable field and estimate the peak-peak correlation function for a Gaussian field. In Section 3 we describe the general treatment to obtain the peak-peak non-Gaussian correlation function. In Section 4 we apply the calculations for a slightly non-Gaussian model in two steps: first we consider an approximated solution for a non-Gaussian model with null high order correlations and finally we estimate the complete solution for a slightly non-Gaussian field. In section 5, we summarise our results and discuss the possibilities to use the peak-peak correlation function as a statistical descriptor of the density fluctuation field.

2. Random Variable Fields

Most of the models for the early universe (i.e. inflation) actually predicts the fluctuation field to be random. This requires that $\delta(\mathbf{x})$ can be treated as a random variable in the 3D-space and the assumption that the universe is a random realization from a statistical ensemble of possible universes.

We define a random variable, δ, using the fact that, instead of knowing its exact value, we only know how to measure various values of δ_1, δ_2, ... δ_n, which define a random variable field, under certain experimental conditions. Therefore, a random variable can only be characterized by a certain statistical ensemble of realizations. When we say that a random variable is known, it means that we only know the statistical sample which characterizes it. To completely describe the statistical properties of a random variable, δ, we define the probability density function, $P[\delta]$, which can be obtained from the Fourier transform of the characteristic distribution function, Θ_δ (Gnedenko & Kolmogorov 1968):

$$P[\delta] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\delta}\Theta_\delta(u)du \tag{1}$$

The characteristic function, $\Theta_\delta(u)$, can be obtained by the McLaurin series for the moments, m:

$$\Theta_\delta(u) = 1 + \sum_{n=1}^{\infty}\frac{(iu)^n}{n!}m_n, \quad for: m_n = \langle\delta^n\rangle \tag{2}$$

Another possibility to obtain the characteristic function of a random variable is to use the distribution in series of cumulants:

$$\Theta_\delta(u) = exp\left[\sum_{n=1}^{\infty}\frac{(iu)^n}{n!}k_n\right], \quad for: k_n = \langle(\delta - <\delta>)^n\rangle \tag{3}$$

Since the physical importance of the cumulants k_n in Eq. 3 decreases as n increases, it is usual to confine the statistical calculations of random variables to the first few terms of the cumulant distribution series and, for convenience, set the higher order terms to zero. However, the calculations presented in the next sections show that, even if the cumulants terms are very small, they can significantly contribute to the statistical description of non-Gaussian fields.

2.1. Correlations in a Random Field

The main numerical indication of the correlation degree between random variables are the N-order correlation functions. The autocorrelation (or double correlation) for a random variable δ is defined by:

$$K_2[\delta_1\delta_2] = \langle\delta_1\delta_2\rangle - \langle\delta_1\rangle\langle\delta_2\rangle \tag{4}$$

The triple correlation is similarly defined in terms of all possible combinations between the three variables, being:

$$\begin{aligned} K_3[\delta_1\delta_2\delta_3] \;&= \langle\delta_1\delta_2\delta_3\rangle - \langle\delta_1\rangle K[\delta_2\delta_3] - \langle\delta_2\rangle K[\delta_1\delta_3] \\ &- \langle\delta_3\rangle K[\delta_1\delta_2] - \langle\delta 1\rangle\langle\delta 2\rangle\langle\delta 3\rangle \end{aligned} \tag{5}$$

Note that, in the case where we have the same three variables ($\delta_1 = \delta_2 = \delta_3$), the three-point correlation function is similar to the third cumulant of the distribution. Higher order correlations between several variables can be defined, in a similar way, by the difference between all possible correlations involved.

For a statistical process where the correlation functions of order greater than one are null, we set the variable described by this correlation function as not random, or deterministic. In the case where the correlation functions of order greater than two are null, we have a Gaussian variable. For the case of correlation functions of order greater than two not completely null, the variable is considered to be non-Gaussian. In this sense, we can say that a Gaussian random field is a simplified version of a general random field.

Usually, the cosmological density fluctuation field is statistically described by the mean correlation function, $\xi(r)$, applied to a galaxy or a cluster distribution, with two-point mean separation defined by r (Peebles 1980). For an isotropic and homogenous field, the correlation function is defined as the excess of probability for a density field described by a Poisson distribution. Therefore, the probability to find two points, in a volume $dV_1 dV_2$, separated by a distance r_{12} is given by:

$$dP = n^2 dV_1 dV_2 \left[1 + \xi(r_{12})\right] \tag{6}$$

Describing the fluctuations field in Fourier modes, we have:

$$\xi(r) \equiv \left\langle \sum_k \sum_{k'} \delta_k \delta_{k'} e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot \boldsymbol{r}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \right\rangle, \tag{7}$$

which is equivalent, in a continuous space, to:

$$\xi(r) = \frac{V}{(2\pi)^3} \int |\delta_k|^2 \, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d^3\boldsymbol{k} \tag{8}$$

2.2. High Density Peaks in a Gaussian Random Field

For a Gaussian random field, the n-dimensional probability density function can be estimated from the Fourier transform of the characteristic function (Eq. 1 and 2) defined for moments distribution of s ≤ 2.

$$P^G[\delta_1, \delta_2 ... \delta_n] = \tfrac{1}{(2\pi)^n} \int_0^\infty du_1 \int_0^\infty du_2 ... \int_0^\infty du_n$$
$$exp\left[\sum_{s=1}^{2} \frac{i^s}{s!} \sum_{\alpha,\beta=1}^{n} K_s(\delta_\alpha, \delta_\beta) u_\alpha u_\beta \right] exp\left[-i \sum_{\alpha=1}^{n} u_\alpha \delta_\alpha\right] \tag{9}$$

The expression above can be reduced to:

$$P^G[\delta_1, \delta_2 ... \delta_n] \;=\; \tfrac{1}{(2\pi)^{\frac{n}{2}}} \left[det\|K_2(\delta_\alpha, \delta_\beta)\|\right]^{-\frac{1}{2}}$$
$$exp\left[-\tfrac{1}{2} \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta} \left[\delta_\alpha - K_1(\delta_\alpha)\right] \left[\delta_\beta - K_1(\delta_\beta)\right]\right], \tag{10}$$

where $det\|K_2(\delta_\alpha, \delta_\beta)\|$ and $\|a_{\alpha\beta}\|$ are, respectively, the determinant and the inverse of the correlation matrix.

For a bidimensional Gaussian fluctuation field with zero mean, the correlation matrix can be obtained and inverted, resulting in:

$$P^G[\delta_1, \delta_2] = \frac{1}{2\pi\sigma^2} \left(\frac{1}{1 - A^2}\right)^{\frac{1}{2}} e^{-Q[\delta_1, \delta_2]}, \tag{11}$$

where $A = \frac{\xi_r}{\sigma^2}$, for:

$\xi_r = \langle \delta_1 \delta_2 \rangle$, $r = |\boldsymbol{x}_2 - \boldsymbol{x}_1|$, $\sigma^2 = \left\langle \delta_1^2 \right\rangle = \left\langle \delta_2^2 \right\rangle$, and:

$$Q = \frac{1}{2} \left(\frac{1}{1 - A^2}\right) \frac{1}{\sigma^2} \left[\delta_1^2 + \delta_2^2 - 2A\delta_1\delta_2\right], \tag{12}$$

where ξ_r is equal to $K_2(\delta_1, \delta_2)$, the mean correlation function for a Gaussian field.

To find the correlation function between high density peaks, we calculate the probability of δ_1 and δ_2 to exhibit density values which are larger than the variance field σ, by a factor $\eta (\eta > 0)$. For a Gaussian field, this probability is given by the integral (Padmanabhan 1999):

$$P^G[\delta_1 > \eta\sigma, \delta_2 > \eta\sigma] = \int_{\eta\sigma}^{\infty} d\delta_1 \int_{\eta\sigma}^{\infty} d\delta_2 P^G[\delta_1, \delta_2]. \tag{13}$$

The integral above can be obtained by substituting Eq. 9 into Eq. 13. For a weakly correlated field, where $A = \frac{\xi_r}{\sigma^2} << 1$, and high density peaks, $\eta > 1$, the integral above will be:

$$P^G[\delta_1 > \eta\sigma, \delta_2 > \eta\sigma] =$$
$$\int_{\eta\sigma}^{\infty} d\delta_1 \int_{\eta\sigma}^{\infty} d\delta_2 \left[\tfrac{1}{\sigma^2} \sum_{p=0}^{1} F^{p+1}\left(\frac{\delta_1}{\sigma}\right) F^{p+1}\left(\frac{\delta_2}{\sigma}\right)\right] \tfrac{(2A)^p}{p!}, \tag{14}$$

where F is the Err function (erf):

$$F^{p+1}(Z) = \frac{(-i)^p}{2\pi} \int_{-\infty}^{\infty} \lambda^p e^{\left[-i\lambda z - \frac{z^2}{2}\right]} d\lambda. \tag{15}$$

The final result is:

$$P^G[\delta_1 > \eta\sigma, \delta_2 > \eta\sigma] = \left[\frac{1}{\sqrt{2\pi}} \frac{1}{\eta} e^{-\frac{\eta^2}{2}}\right]^2 (1 + 2A\eta^2). \tag{16}$$

Redefining expression 16, the probability to find peaks δ_1 and δ_2 with density η times the variance, σ, is:

$$P^G[\delta_1 > \eta\sigma, \delta_2 > \eta\sigma] \equiv [P^G(\delta > \eta\sigma)]^2 [1 + \xi_\eta^G(r)], \tag{17}$$

where $[P^G[\delta > \eta\sigma)]]^2$ is the mean probability to find high density peaks in a bi-dimensional random field, and $[1 + \xi_\eta^G(r)]$ is the probability excess expressed in terms of the mean correlation function, ξ_r. Then, for weakly correlated fields:

$$\xi_\eta^G(r) \equiv 2A\eta^2 \approx e^{2A\eta^2} - 1 = exp\left[2\left(\frac{\eta}{\sigma}\right)^2 \xi_r\right] - 1. \tag{18}$$

3. Correlations in a Generic Non-Gaussian Field

The correlation function of high density peaks in a non-Gaussian field can also be computed using Eq. 13, except that we have to consider the non-Gaussian probability of finding both peaks δ_1 and δ_2, the $P^{NG}[\delta_1, \delta_2]$. This probability can be estimated using Eq. 9, performing the sum over terms of order higher than 2. Since the summation is also computed for additional terms, the result can be synthesized by:

$$P^{NG}[\delta_1 > \eta\sigma, \delta_2 > \eta\sigma] = [P^G(\delta > \eta\sigma)]^2 \left[1 + \xi_\eta^G(r) + C_\eta^{NG}\right], \quad (19)$$

which results in:

$$P^{NG}[\delta_1 > \eta\sigma, \delta_2 > \eta\sigma] = \left[\frac{1}{\sqrt{2\pi}}\frac{1}{\eta}e^{-\frac{\eta^2}{2}}\right]^2 (1 + 2A\eta^2 + C_\eta^{NG}), \quad (20)$$

where the factor C_η^{NG} carries the corrections terms of higher order (s ≥ 3).

One intriguing question we could ask is how important the high order residual terms for a slightly non-Gaussian statistics are . First, we could consider the approximated case in which the extra calculation in Eq. 9 were avoided, so correlation terms of order (s > 2) could be neglected ($C_\eta^{NG} \sim 0$) and the non-Gaussian peak-peak correlation function would be reduced to:

$$\xi_\eta^{NG}(r) \approx exp\left[2\left(\frac{\eta}{\sigma}\right)^2 \xi_r^{NG}\right] - 1, \quad (21)$$

This solution is similar to the calculation for a Gaussian random field, except for the fact that we were considering a non-Gaussian mean correlation function.

At this point we want to assess the robustness of the assumption $C_\eta^{NG} \sim 0$, stated in the previous paragraph. For this purpose, we compute the higher order correction terms considering a slightly non-Gaussian component, this means a very small contribution to correlations of order greater than two. However, the non-Gaussian probability described in Eq. 9 can not be calculated using the Fourier transform of the characteristic function, since there is no analytical solution for that expression. One possible way to obtain $P^{NG}[\delta_1, \delta_2]$, as suggested by (Gnedenko & Kolmogorov 1968), is to expand it in a series of Generalized Hermite polynomials, H:

$$P^{NG}[\delta_1, \delta_2 ... \delta_n] = P^G[\delta_1, \delta_2 ... \delta_n] \left[1 + \sum_{s=3}^{\infty} \frac{1}{s!} \sum_{\alpha,\beta...\omega=1}^{n} b_s(\delta_\alpha, \delta_\beta ... \delta_\omega) H_{\alpha,\beta...\omega}\left[\delta - \kappa_1(\delta)\right]\right], \quad (22)$$

where b_s is the quasi-moment function and κ_1 is the first cumulant of the distribution. The definitions of b_s and H are given in Appendix A.

By Eq. 22, the non-Gaussian probability is expressed in terms of a Gaussian probability added to higher order ($s \geq 3$) correction terms, which are related to the deviation from Gaussianity. Combining equations 13, 19 and 22,we can estimate the higher order corrections term for a bi-dimensional using the following:

$$C_\eta^{NG} = \int_{\eta\sigma}^{\infty}\int_{\eta\sigma}^{\infty} P^G[\delta_1, \delta_2] d\delta_1 d\delta_2 \left[\sum_{s=3}^{\infty} \frac{1}{s!} \sum_{\alpha,\beta...\omega=1}^{2} b_s(\delta_\alpha, \delta_\beta) H_{\alpha,\beta}\left[\delta - \kappa_1(\delta)\right]\right]. \quad (23)$$

Note that the calculation of C_η^{NG} starts at s ≥ 3, ensuring that the terms used in the expansion are related to the non-Gaussian contributions. Details of the calculation involved in this higher order terms are also presented in Appendix A.

4. Correlations in a Nearly-Gaussian Field

4.1. The Mixture Model

In order to estimate how the high order terms affect the correlation between high density peaks, we estimated the ξ_η for a Gaussian and a slightly non-Gaussian field, computing the approximated solutions (Eq. 21) and the full calculation of the expansion (Eq. 22) until $s \leq 6$ order. For this comparison, we have considered a mixed probability distribution, as proposed by Ribeiro, Wuensche & Letelier (2001), hereafter (RWL).

The general procedure to create a wide class of non-Gaussian models is to admit the existence of an operator which transforms Gaussianity into non-Gaussianity according to a specific rule. An alternative approach to study non-Gaussian fields was proposed by RWL, in which the PDF is treated as a mixture: $P(\psi) = (1 - \alpha) f_1(\phi) + \alpha f_2(\phi)$, where $f_1(\phi)$ is a (dominant) Gaussian PDF and $f_2(\phi)$ is a second distribution, with $0 \leq \alpha \leq 1$. The α parameter gives the absolute level of Gaussian deviation, while $f_2(\phi)$ modulates the shape of the resultant non-Gaussian distribution. RWL used this mixed scenario to probe the evolution of galaxy cluster abundance in the universe and found that even at a small level of non-Gaussianity ($\alpha \sim 10^{-4} - 10^{-3}$) the mixed field can introduce significant changes in the cluster abundance rate.

The effects of such mixed models in the CMBR power spectrum, combining a Gaussian adiabatic field with a second, non-Gaussian isocurvature fluctuation field, to produce a positive skewness density field was discussed by Andrade, Wuensche & Ribeiro (2004) (hereafter AWR04). In this approach, they adopted a scale dependent mixture parameter and a power-law initial spectrum to simulate the CMBR temperature and polarization power spectra for a flat Λ-CDM model, generating a large grid of cosmological parameters combination. The choice of a scale-dependent mixture is not unjustified, since it could fit both CMB and high-z galaxy clustering in the Universe (e.g. AWR04; Avelino & Liddle 2004; Mathis, Diego & Silk 2004). At the same time, in mixed scenario, the scale-dependence acts in order to keep a continuous mixed field inside the Hubble horizon. Simulation results show how the shape and amplitude of the fluctuations in CMBR depend upon such mixed fields and how a standard adiabatic Gaussian field can be distinguished from a mixed non-Gaussian one. They also allow one to quantify the contribution of the second component. By applying a χ^2 test on recent CMBR observations, the contribution of the isocurvature field was estimated by Andrade, Wuensche & Ribeiro (2005) (hereafter AWR05) as α_0= 0.0004 ± 0.00030 with 68% confidence limit. In the present work, we also investigate the predictions of scale-dependent mixed non-Gaussian cosmological density fields for the peak-peak correlation function.

4.2. The Two-Point Correlation Function

To obtain the mean correlation function, ξ_r, we have computed the Fourier transform of the power spectrum related to a pure Gaussian field and a mixed non-Gaussian PDF. In this sense, we rewrite Eq. 8, which is equivalent to:

$$\xi(r) = \frac{V}{(2\pi)^3}\int_0^\infty P(k)k^2dk\int_0^{2\pi}d\phi\int_{-1}^{1}e^{-ikru}du, \qquad (24)$$

being $u = cos\theta$. For an isotropic field, we have:

$$\xi(r) = \frac{V}{(2\pi)^3}\int_0^\infty P^{NG}(k)\frac{sin(kr)}{kr}k^2dk \qquad (25)$$

For the mixture correlation function, we consider a mixed-scale power spectrum described as:

$$P^{Mix}(k) = A_nP(k)[1 + M(\alpha_0)k] = \beta k^n[1 + M(\alpha_0)k] \qquad (26)$$

This power spectrum was estimated by the correlated mixed-model which considers a possible mixture, inside the horizon, between fluctuations of an adiabatic Gaussian field and an isocurvature non-Gaussian one (AWR04). In this model, α_0 is the mixture parameter that modulates the contribution of the isocurvature field (α_0 = (0.00042 ± 0.00030)) as estimated from recent CMB observations. For a null α_0, the field is purely Gaussian with a simple power law spectrum. A_n is the normalization constant of the primordial power spectrum, estimated as $\sim 1.3\times10^{-11}$ for recent CMBR observations (AWR05). $M(\alpha_0)$ is the mixed term, which accounts for the statistical effects of the second component in the power spectrum, expressed as a functional of both distributions, f_1 and f_2:

$$M^{mix}(\alpha_0, k) \equiv \int_\nu[(1-\alpha_0k)f_1(\nu) + \alpha_0kf_2(\nu)]\nu^2\,d\nu \qquad (27)$$

Evaluating the integral in Eq. 25 for a mixed power spectrum, we find an expression for the mean correlation function:

$$\xi^{Mix}(R) = A_n\left[\frac{1.76\times10^{-2}}{3\pi\left(\frac{R}{R_0}\right)} + \frac{5.76\times10^{-4}M(\alpha_0)}{3\pi\left(\frac{R}{R_0}\right)^2}\right], \qquad (28)$$

where R_0 is about 25 Mpc, the mean correlation width for galaxy clusters.

In Figure 1, we show the mean correlation function estimated for a pure Gaussian field, $M(\alpha_0) = 0$, for a mixed PDF, $M(\alpha_0) = \alpha_0\left(\frac{e^2}{2}-1\right)$, and also the individual contribution of the non-Gaussian field for $\alpha_0 \sim 10^{-3}$. In this plot, it is possible to observe the importance of even a small contribution of a the second component to the mean correlation function. For $\frac{R}{R_0} \le 1\times10^{-3}$ the non-Gaussian component dominates the correlation function. This behaviour illustrates the excess of power in small scales, as observed in the CMB angular power spectrum in the mixed context (AWR04).

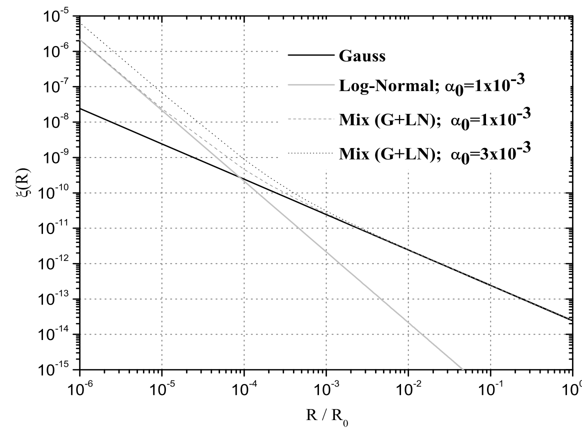


Fig. 1. The mean two-point correlation function estimated for non-correlated or weakly correlated fields: pure Gaussian, pure log-normal and Mixed field with $\alpha_0 \sim 10^{-3}$.

Inserting the mean correlation function described in Eq. 28 into the approximated expression of the correlation function for high density peaks (Eq. 21) for a few classes of PDFs, we estimate the functions ξ_η plotted in Figure 2. In this plot, we show that the effect of the second component is still observed, and that the correlation function for high density peaks is also sensible for different non-Gaussian distributions. Comparing Figures 1 and 2, we see how the high density peaks can be much more correlated than the mean field for a non-Gaussian case, especially in small scales, $\frac{R}{R_0} \sim 10^{-5}$ where ξ_η^{Mix} is nearly two orders of magnitude greater than the mean correlation.

With the help of a program that performs algebraic and numerical calculations, we actually computed C_η^{NG}, as indicated in Eq. 23, for correlations up to the 6-th order ($s \le 6$). This limit was set in order to keep a meaningful non-Gaussian distortion, avoiding more time consuming calculations. We do not present in this section the full computed expression since it contains hundreds of non-linear terms in the mean correlation function (ξ_r) and η, where the coefficients are the high order correlations (or cumulants). In fact, the C_η^{NG} is more accurately described as $C_\eta^{NG}(\xi_r)$. In Appendix A, we show the steps to calculate the quasi-moment function, b_{lm}, and the Hermite Polynomials, H_{lm}.

In general, correlations of very high order tend to zero (Gnedenko & Kolmogorov 1968), the most extreme case being the normal distribution, where all cumulants of $s \ge 3$ are null. Deviations from Gaussianity are set by the increment of non-vanishing cumulants in the expansion of C_η^{NG}. A possible question that may be raised is related to the convergence and nor-

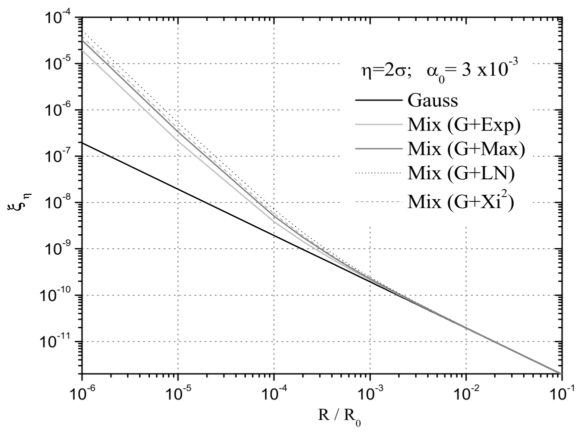


Fig. 2. The two-point correlation function computed for 2σ density peaks in both a pure Gaussian and mixed context ($\alpha_0 \sim 10^{-3}$). For this estimation, we used: $M^{Gaus}(\alpha_0) = 0$; $M^{Exp}(\alpha_0) = \alpha_0$; $M^{Max}(\alpha_0) = \alpha_0\left(3\sqrt{\frac{\pi}{2}} - 1\right)$; $M^{Xi^2}(\alpha_0) = 2\alpha_0$ and $M^{LN}(\alpha_0) = \alpha_0\left(\frac{e^2}{2} - 1\right)$.

malization of the expansion in Eq. 23 in which some terms are set as non-vanishing. However, investigation of general nearly gaussian deviations have already been performed by the use of the Edgeworth expansion in one-dimension by setting the function to zero in higher orders terms and normalizing it appropriately (Martinez-Gonzalez et al. 2002). Following these authors, we have considered, as a work hypothesis, the following behavior for such coefficients: we have set the terms $s \geq 7$ to zero and the cumulants involved in the quasi moment function to 10^{-3} for $3 \leq s \leq 6$.

A very interesting result obtained is summarized in Figure 3. For this plot, in order to follow the absolute behavior of the corrections terms in the nearly Gaussian field, we have set the normalization value of the mean correlation function to one, ($\xi_r \sim 1$) and explore the η dependence of $C_\eta^{NG}(\xi_r)$ with the purpose to understand how the amplitude of the fluctuations is related with the high order correlations. While computing only third-order terms, the C_η^3 describes a gradual enhancement in the correlation function. For fourth and sixth-order terms, the C_η^4 and C_η^5, in the case of small Gaussian deviation, what is observed is that the C_η^4 and C_η^5 just overlap each other, meaning that the fifth order correlations do not contribute significantly. However, the fourth order correlation describes peaks of maximum density allowed by a weakly correlated field at about $\eta \sim 1.8\ \sigma$. For higher order correlations, the C_η^6 shows a very large increment in the two point correlation for densities about $2.5\ \sigma$, and a nearly null contribution to peaks with amplitudes ($\eta \geq 3\sigma$).

In this sense, we conclude that one should not expect very high density peaks for the specified, slightly non-Gaussian, correlated field. However, factors or order $3 \leq s \leq 6$, are quite significant for peaks with amplitude up to $2.5\ \sigma$ and are too far from a null correction, even if we consider weakly correlated fields. Then, we can not consider the simplified solution in Eq. 21 as a good description of the two-point correlation function for high density peaks in despite of our choice of considering just a small deviation from Gaussianity. It is important to note that this result is independent of the power spectrum or the mean correlation function. It only shows the influence of higher order correlations in the amplitude of the field of fluctuations.

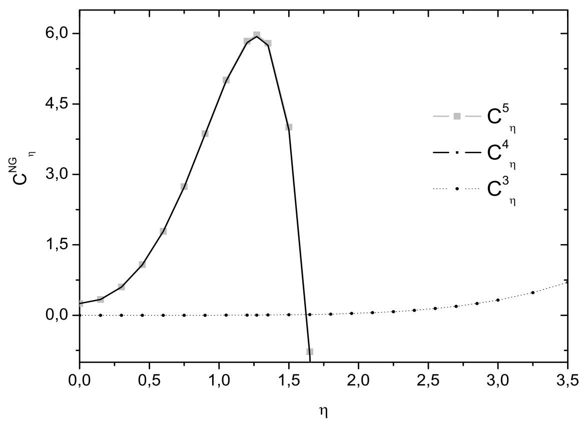
Analyzing the relation between C_η and η we gain some insight about the amplitude of such high order terms, since C_η controls the amplitude of the permitted peaks in the fluctuations field. Increasing values of correlations with $s > 6$ imply a high probability of very high amplitude (very rare) peaks, which contradict the observations of large scale structures. However, when we impose correlation levels of the order 10^{-3} up to sixth order, we favour the existence of peaks up to 3σ, what is very reasonable for a nearly Gaussian field.

In Figure 4 we show the behaviour of the two-point correlation function estimated for a Gaussian, ξ_η^{Gauss}; for a mixed approximated solution, ξ_η^{Mix}; and for the non-Gaussian complete solution, $\xi_\eta^{Mix} + C_\eta^{NG}(\xi_r)$. In this plot, we have set the amplitude threshold of $\eta \sim 2$ and test the dependence on ξ_r. The observed effect of $C_\eta^{NG}(\xi_r)$ is to amplify the correlations between high density peaks. This result is valid for the case of an increasing mean correlation function, and has no dependence of the mixed model. While non-Gaussian deviation tends to add non-vanish high order correlations, we conclude that we can expect high density peaks to be much more correlated even in a slightly non-Gaussian model.

5. Discussion

In this work, we have estimated the two-point mean correlation function and the peak-peak correlation function for the density field. In the Gaussian case, the calculations are simplified, since the Fourier transform of the power spectrum completely describes the random variable. However, for a non-Gaussian field, the calculations are much more complicated, since high order correlations between these two-points may not vanish and strongly contribute to the final function, even in a small deviation from the Gaussian case.

In this work, we showed that, when considering a mixed model, both the mean correlation and the peak-peak correlation functions are much more intense in small scales than in the Gaussian case. This result can be particularly relevant since it is generally accepted that galaxies form in high density regions. In addition, we conclude that the peak-peak correlation function is quite sensitive to the PDF of the fluctuations field, especially for a mixed model. This result suggests that it is possible to use the peak-peak correlation function as a test for the nature of cosmic structures. Nevertheless, we have to be careful when approximating terms for high order correlations, since the peak-peak correlation function is very sensitive to the correction terms. We also point out that correlations of order > 2 can be a very important tool to characterize non-Gaussian fields and definitely deserve a deeper investigation. Estimating high order correlations allows us to investigate the behaviour of the N-point correlation function, as well as gather more information about the amplitude of the expected high density fluctuation field. As seen in Figure 3, correlations may restrict the



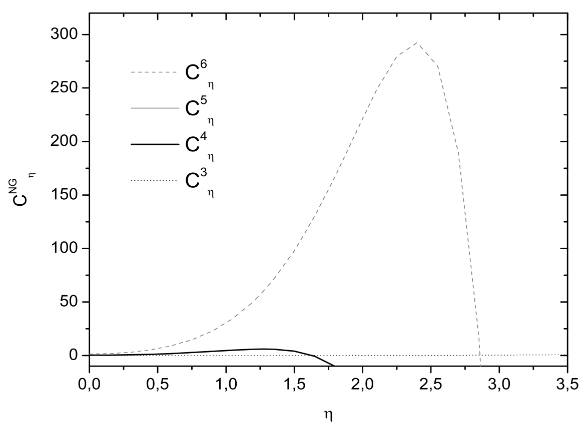


Fig. 3. The behaviour of the higher order factor, C_η^{NG}, for a fixed mean correlation ($\xi_r \sim 1$). The curves in A) show the factor estimated for third order correlation, C_η^3, fourth, C_η^4, and fifth, C_η^5. In B) the sixth order factor, C_η^6, is also plotted.

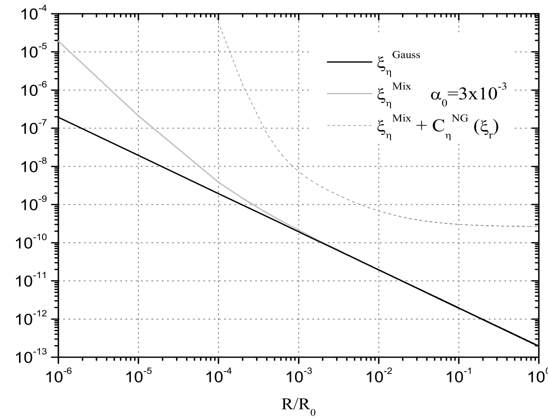


Fig. 4. The behaviour of the two-point correlation function is shown for three estimated cases: a pure Gaussian, ξ_η^{Gauss} (lower curve, orange), a mixed approximated solution, ξ_η^{Mix} with $\alpha_0 \sim 3 \times 10^{-3}$ (mid curve, black); and for the complete non-Gaussian solution, $\xi_\eta^{Mix} + C_\eta^{NG}(\xi_r)$, estimated for a PDF of the type: (Gauss + Exp) with $\alpha_0 \sim 3 \times 10^{-3}$

amplitude of the density peaks. Furthermore, from the amplitude of the peaks found in the density field (in CMB or LSS fluctuations) we are able to extract more information about the statistics of the density field. It is also good to remember that the presence of correlations of order $s \geq 3$ lead to formation of structures in earlier times than would be expected for a model with the same power spectrum but with weaker spatial correlations. The resuls presented in this paper may be used to set new constraints in structure formation models.

One possible application of this method on investigations of primordial non-Gaussianity could be implemented in the search for maximum amplitude fluctuations of the full sky CMB temperature maps, such those derived from WMAP (Spergel et al. 1989) and the future Planck mission (Wright E.L. 2000). However, variations in the number of density peaks and their correlation may as well be related to non-Gaussian Galactic foregrounds or other contaminants. Once detected such non-Gaussian trace, we have to be very careful before assigning it to a primordial origin. To avoid misunderstanding, the investigation of the peaks statistics in CMB datasets should be realized over several datasets and different frequencies. Indeed, one possible manner to minimize the foregrounds effects is to analyse the most sensitive and cleaned map, such the WMAP three-year di-biased internal linear combination, WMAP-DILC map, or the WMAP coadded map, with the combination of Q+V+W frequencies. This analysis will be the next test applied to the non-Gaussian mixed model.

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Appendix A: Calculation of b_s and H

To obtain the non-Gaussian probability for a multi-dimensional case we have to perform the calculation in Eq. 22 expanding the Hermite Polynomials and the quasi-moment function. One possible way to obtain the quasi-moment functions, b_s, is relating to the correlation function, since:

$$exp\left[\sum_{s=3}^{\infty}\sum_{\alpha,\beta,...,\omega=1}^{n} K_s(x_\alpha, x_\beta, ..., x_\omega) z_\alpha z_\beta ... z_\omega\right]$$
$$= 1 + \sum_{s=3}^{\infty}\frac{i^s}{s!}\sum_{\alpha,\beta,...,\omega=1}^{n} b_s(x_\alpha, x_\beta, ..., x_\omega) z_\alpha z_\beta ... z_\omega. \qquad (A.1)$$

The Generalized Hermite polynomials can be obtained by the definition:

$$H_{\alpha\beta...\omega}[x] = e^{\phi[x]}\left(-\frac{\partial}{\partial x_\alpha}\right)\left(-\frac{\partial}{\partial x_\beta}\right)...\left(-\frac{\partial}{\partial x_\omega}\right)e^{-\phi[x]}, \qquad (A.2)$$

where: $\phi[x] = \frac{1}{2}\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}x_\alpha x_\beta$, being $||a_{\alpha\beta}||$ the elements of the inverse correlation matrix $||k_2(t_\alpha, t_\beta)||$. The unidimensional case of Eq. 24 is known as Edgeworth series, while the bidimensional case corresponds to:

$$P^{NG}[\delta_1, \delta_2] =$$
$$\left[1 + \sum_{s=3}^{\infty}\frac{1}{s!}\sum_{l+m=s} b_{lm}H_{lm}\left[\delta - \kappa_1(\delta)\right]\right]P^G\left[\delta_1, \delta_2\right], \qquad (A.3)$$

where: $b_{lm} = b_{l+m}[\underbrace{\delta_1\cdots\delta_1}_{l\text{ times}};\underbrace{\delta_2\cdots\delta_2}_{m\text{ times}}]$,

and: $H_{lm} = H_{\underbrace{1\cdots1}_{l\text{ times}};\underbrace{2\cdots2}_{m\text{ times}}}$.

Performing the calculations for the quasi-moment function we have:

$$b_3 = K[\delta_1, \delta_2, \delta_3];$$
$$b_4 = K[\delta_1, \delta_2, \delta_3, \delta_4];$$
$$b_5 = K[\delta_1, \delta_2, \delta_3, \delta_4, \delta_5];$$
$$b_6 = K[\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6] + 10\left[K_3[\delta_1, \delta_2, \delta_3]K_4[\delta_4, \delta_5, \delta_6]\right].$$

Note that the first five terms of b_s are equivalent to the correlation function, only the sixth term have additional terms. As an example, for $s = 3$, we obtain four different forms of b_3:

$$b_{30} = K_3[\delta_1, \delta_1, \delta_1,] = \left\langle\delta_1^3\right\rangle;$$
$$b_{03} = K_3[\delta_2, \delta_2, \delta_2,] = \left\langle\delta_2^3\right\rangle;$$
$$b_{21} = K_3[\delta_1, \delta_1, \delta_2,] = \left\langle\delta_1^2\delta_2\right\rangle;$$
$$b_{12} = K_3[\delta_1, \delta_2, \delta_2,] = \left\langle\delta_1\delta_2^2\right\rangle.$$

Proceeding in a similar manner, we find five terms for b_4, six for b_5 and seven terms for b_6.

The best way to find the Hermite Polynomials is using expression (Gnedenko & Kolmogorov 1968):

$$y_\alpha = y_\alpha[x] = \frac{\partial\phi[x]}{\partial x_\alpha} = \sum_{\beta=1}^{n} a_{\alpha\beta}x_\beta, \quad being: \qquad (A.4)$$

$$
\begin{aligned}
H_{\alpha} &= y_{\alpha}, \\
H_{\alpha\beta} &= y_{\alpha}y_{\beta} - a_{\alpha\beta}, \\
H_{\alpha\beta\gamma} &= y_{\alpha}y_{\beta}y_{\gamma} - a_{\alpha\beta}y_{\gamma} - a_{\alpha\gamma}y_{\beta} - a_{\gamma\beta}y_{\alpha} \\
&= y_{\alpha}y_{\beta}y_{\gamma} - \left\{a_{\alpha\beta}y_{\gamma}\right\}_3, \\
H_{\alpha\beta\gamma\delta} &= y_{\alpha}y_{\beta}y_{\gamma}y_{\delta} - \left\{a_{\alpha\beta}y_{\gamma}y_{\delta}\right\}_6 - \left\{a_{\alpha\beta}a_{\gamma\delta}\right\}_3, \\
H_{\alpha\beta\gamma\delta\omega} &= y_{\alpha}y_{\beta}y_{\gamma}y_{\delta}y_{\omega} - \left\{a_{\alpha\beta}y_{\gamma}y_{\delta}y_{\omega}\right\}_{10} - \left\{a_{\alpha\beta}a_{\gamma\delta}y_{\omega}\right\}_{15}, \\
H_{\alpha\beta\gamma\delta\omega\theta} &= y_{\alpha}y_{\beta}y_{\gamma}y_{\delta}y_{\omega}y_{\theta} - \left\{a_{\alpha\beta}y_{\gamma}y_{\delta}y_{\omega}y_{\theta}\right\}_{15} - \\
&\quad \left\{a_{\alpha\beta}a_{\gamma\delta}y_{\omega}y_{\theta}\right\}_{45} - \left\{a_{\alpha\beta}a_{\gamma\delta}a_{\omega\theta}\right\}_{15}.
\end{aligned}
$$

where we have used the notation $\{i\}_s$ to indicate a simetrization set.

To perform the calculation above for b_{lm} and H_{lm} up to $(l + m = 6)$ and perform the integration in $d\delta_1$ and $d\delta_2$, we used a software for algebraic and numerical calculations.